SM311O Second Exam
26 Mar 1999

1. (**30 points**) Let $\psi = 2x^2 + 3y^2 - xy$ be the stream function of a flow.

   (a) Determine the velocity field associated with ψ.

   (b) Determine the line integral of the velocity field along the straight line that connects the points $(-1, 2)$ to $(2, -1)$.

   (c) Determine the circulation of this flow around a circle of radius 3 centered at the origin.

2. (**15 points**) Let $\mathbf{v} = \langle 6x^2 + z, -z^2, x - 2yz \rangle$.

   (a) Determine whether this velocity field has a potential ϕ. If yes, find ϕ.

   (b) Determine the line integral of this flow along the parabola $y = x^2$ in the $z = 1$ plane from A to B where $A = (0, 0, 1)$ and $B = (2, 4, 1)$.

3. (**15 points**) Let $f(x) = x(1-x)$. Find the Fourier sine series of f in the interval $(0, 3)$. Use the first nonzero term of the Fourier series and evaluate it at $x = \frac{1}{2}$. How much does this value differ from $f(\frac{1}{2})$?

4. (**25 points**) Consider the wave equation initial–boundary value problem

$$u_{tt} = 4u_{xx}, \quad u(x, 0) = x(1-x), \quad u_t(x, 0) = 0, \quad u(0, t) = u(3, t) = 0.$$

   (a) Explain in words what $u(t, x)$ and each term in the above equations represent.

   (b) Determine the solution to this problem (you may wish to use the result of your computations in the previous problem).

   (c) Using only one term of the series solution in part (a), determine how long it takes for the string to go through one oscillation.

5. (**15 points**) Consider an incompressible fluid occupying the slab

$$D = \{(x, y, z) | 0 \le z \le H\}.$$

   Let $\mathbf{v} = \langle v_1(x, y, z), v_2(x, y, z), v_3(x, y, z) \rangle$ be the velocity field of a motion generated in D. Suppose that

$$v_1 = x^2y^2z - xy, \quad v_2 = 3x^2 + y^2,$$

everywhere in D and that the vertical component of the velocity, v_3, is measured to be

$$x + y$$

at the **bottom** of D, i.e., when $z = 0$. Determine v_3 everywhere in D. (Hint: What does incompressibility mean **mathematically**?)